$$g_k = KLN_1\mu^{a-1}[d_k(a_k + c_k) + b_k(1 + \mu^2 e_k)]$$
$$d_k = \nu(\xi_k) + \nu(\xi_{k-1}) + \nu(y_k) + \nu(y_{k-1}) + \nu(z_k) + \nu(z_{k-1}) + \mu^{-2}$$
$$e_k = \nu(y_k) + \nu(y_{k-1}) \quad (k = 1, 2, \ldots)$$

Estimating d_k and e_k by means of inequalities (4.4) we obtain

$$a_{k+1} \leqslant P_1\mu^a(\mu^{a-1}a_k + b_k + c_k), \quad b_{k+1} \leqslant g_k', \quad c_{k+1} \leqslant \mu g_k'$$
$$g_k' = P_1\mu^{a-1}[\mu^{a-1}(a_k + c_k) + b_k], \quad P_1 = KL(2B_3 + 4B_4 + 1) \times \max(N_0, N_1)$$

Consider the sequence of numbers $\rho_k = \mu^{(a-3)/2}a_k + b_k + \mu^{-1}c_k$ $(k = 1, 2, \ldots)$. For $\mu \geqslant M = \max[M_2, (6P_1)^{2/(1-3a)}]$ we have $\rho_{k+1} \leqslant \rho_k/2$ $(k = 1, 2, \ldots)$. Using this estimate we can prove that the sequences $\xi_k(t)$, $y_k(t)$, and $z_k(t)$ converge uniformly on the set $\{(t, \mu): 0 \leqslant t \leqslant L\mu^a, \mu \geqslant M\}$ to some continuous functions $\xi_*(t, \mu)$, $y_*(t, \mu)$ and $z_*(t, \mu)$ satisfying the inequalities obtained from (4.4) by the change $\xi_k \to \xi_*$, $y_k \to y_*$, $z_k \to z_*$. Passing to the limit in (4.2) as $k \to \infty$ we find that $\xi_*(t, \mu)$, $y_*(t, \mu)$, and $z_*(t, \mu)$ are the solutions of the system of integral Eqs.(4.1). The function $\xi_*(t, \mu)$ is continuously differentiable in t, the function $y_*(t, \mu)$ is twice continuously differentiable in t, and $y_*^{\cdot}(t, \mu) = z_*(t, \mu)$.

The uniqueness of the solution obtained can be proved in a standard way.

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EQUIVALENT LINEARIZATION OF QUASILINEAR OSCILLATING SYSTEMS WITH SLOWLY VARYING PARAMETERS*

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The problem of the approximate reduction of quasilinear oscillating system with slowly varying parameters to a linear system that is equivalent in the asymptotic sense is investigated /1-3/. Two approaches are proposed based on intermediate "amplitude-phase" variables and osculating variables of the Van der Pol type. An equivalent linear system is also constructed with a prescribed degree of accuracy with respect to a small parameter. As an example a quasilinear oscillator /1-3/ is considered.

The approach developed is based on well-known methods of equivalent linearlization /2-6/ and is interesting from the point of view of applications, since linear equations can be investigated by standard methods. An adequate form of the equations is particularly important in the analysis and synthesis automatic controls systems having the required quality of transients /5-8/.

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1. Statement of the problem of the equivalent linearization of perturbed systems of the gyroscopic type. We consider a quasilinear oscillating system often encountered in mechanics, described by the Cauchy problem of the form /7/

$$x^{\cdot} = \nu(\tau)\, y + \varepsilon f(\tau, x, y), \quad x(0) = x^\circ \tag{1.1}$$
$$y^{\cdot} = -\nu(\tau)\, x + \varepsilon g(\tau, x, y), \quad y(0) = y^\circ$$
$$0 < \nu_1 \leqslant \nu(\tau) \leqslant \nu_2 < \infty, \quad \tau = \varepsilon t + \tau_0;\ \varepsilon \in [0, \varepsilon_0],\ \varepsilon_0 \ll 1$$
$$t \in [0, T(\varepsilon)], \quad T(\varepsilon) = \Theta\varepsilon^{-1}, \quad \Theta = \text{const};\ \tau - \tau_0 \in [0, \Theta]$$

Here x, y are the phase variables belonging to the domain of definition and smoothness of the functions f, g and dots denote derivatives with respect to time t. The function $\nu(\tau)$ has the meaning and dimensions of frequency; in the general case it, and also the functions f, g, may depend on the slowly varying time τ; ε is a small numerical parameter characterizing the small size of the perturbing actions $\varepsilon f, \varepsilon g$ and the rate of the change of the parameters $(\tau^{\cdot} = \varepsilon)$. The constant quantities x°, y°, τ_0 are the initial values of the variables x, y, τ and are assumed to be given and to belong to the domain of definition of system (1.1); Θ is an arbitrarily large but fixed number. In this way the motion is investigated in an asymptotically long (as $\varepsilon \to 0$) but bounded time interval (see below).

Equations of the form (1.1) are often encountered in problems of the dynamics and control of the rotation of a solid /2, 3, 7/. For example, x, y can be projections of the vector of the kinetic momentum or angular velocity on the equatorial plane and ν can be the angular velocity of precession; the dependence of ν on τ is usually called a smooth change of the axial component of these vectors. The perturbing moments εf and εg may be due to small external and internal actions of different physical kinds.

The problem is posed of the equivalent linearization of quasilinear oscillating system (1.1), i.e., a change of variables $x = x(x_1, y_1, \tau, \varepsilon)$, $y = y(x_1, y_1, \tau, \varepsilon)$ such that the Cauchy problem for x_1, y_1 has the form

$$x_1^{\cdot} = p^x(\tau, \varepsilon)\, x_1 + q^y(\tau, \varepsilon)\, y_1, \quad x_1(0) = x_1^\circ(\varepsilon) \tag{1.2}$$
$$y_1^{\cdot} = q^x(\tau, \varepsilon)\, x_1 + p^y(\tau, \varepsilon)\, y_1, \quad y_1(0) = y_1^\circ(\varepsilon)$$

For $\varepsilon = 0$ the homogeneous system (1.2) should degenerate into a system of a simpler form $\xi^{\cdot} = \nu_0\eta$, $\eta^{\cdot} = -\nu_0\xi$, where $\nu_0 = \nu(\tau_0) = \text{const}$ and $\xi(0) = x^\circ, \eta(0) = y^\circ$; the initial system (1.1) degenerates into such a system. In addition we require that the variables x_1, y_1 and x, y, are close in the asymptotic sense: $|x_1 - x| + |y_1 - y| = O(\varepsilon)$ for $t \in [0, T(\varepsilon)]$. The coefficients $p^{x,y}(\tau, \varepsilon)$, $q^{x,y}(\tau, \varepsilon)$ and $x_1^\circ(\varepsilon)$, $y_1^\circ(\varepsilon)$ are unknown and are to be determined.

To solve the problem on a physical level of rigour the methods of optimum (in the sense of the minimum mean square deviation) energy and harmonic linearization were used, see for example /1-6, 8/ (for the principles of "energy" and "harmonic" balances see /2/). An effective mathematical tool for constructing the equivalent linear system (1.2) of definite structure (see below) is the Krylov-Bogolyubov-Mitropol'ski averaging method /1-3/ which we use in this paper. Using asymptotic methods of non-linear mechanics one can obtain a strict mathematical foundation, i.e., one can formulate the requirements for system (1.1) and give an estimate of the error. This approach involves changes of variables, a reduction of the system to a standard form and averaging of the equations /1-3, 7, 9/. After solving them the reverse passage is performed which allows us to obtain the equations of motion in the initial form (1.2).

Before we proceed to construct the linear system (1.2) equivalent to the initial non-linear system (1.1) we make the following remark. A standard oscillating system of the "quasilinear oscillator" type with slowly varying parameters, which is frequently encountered in applications,

$$x^{\cdot\cdot} + \nu^2(\tau)\, x = \varepsilon h(\tau, x, x^\circ), \quad x(0) = x^\circ, \quad x^{\cdot}(0) = x^{\cdot\circ} \tag{1.3}$$

can be reduced to the particular form of system (1.1) by making the change $x^{\cdot} = \nu y$ (the function $\nu(\tau)$ should be differentiable)

$$x^{\cdot} = \nu y, \quad x(0) = x^\circ, \quad y(0) = y^\circ = x^{\cdot\circ}\nu_0^{-1} \tag{1.4}$$
$$y^{\cdot} = -\nu x - \varepsilon\nu'\nu^{-1}y + \varepsilon\nu^{-1}h(\tau, x, \nu(\tau)\, y)$$
$$(f(\tau, x, y) \equiv 0, \quad g(\tau, x, y) \equiv -\nu'\nu^{-1}y + \nu^{-1}h, \quad \nu = \nu(\tau))$$

And conversely, under smoothness conditions on system (1.1), by standard methods of the theory of ordinary differential equations, system (1.1) can be reduced to the form (1.3). In particular, if we differentiate the first equation of (1.1) with respect to t, solve the initial system with respect to the unknown $y, y^{\cdot}$ (for example using the Picard method of successive approximations or the method of tangents) and substitute the expression obtained into the differentiated equation, we obtain an equation of the form (1.3) with respect to x with the function h equal to

$$h = h(\tau, x, x^{\cdot}, \varepsilon) \equiv \nu'(\tau)\, y^* + \nu(\tau)\, g(\tau, x, y^*) + \varepsilon\, f_\tau^* + f_x^*\, x^{\cdot} + f_y^*\, y^{\cdot}$$

$$y = y^*(\tau, x, x^{\cdot}, \varepsilon), \quad y^{\cdot} = y^{\cdot *}(\tau, x, x^{\cdot}, \varepsilon) \equiv -\nu(\tau)\, x + \varepsilon g(\tau, x, y^*(\tau, x, x^{\cdot}, \varepsilon))$$

$$x(0) = x^\circ, \quad x^{\cdot}(0) = x^{\cdot\circ} = \nu(\tau_0)\, y^\circ + \varepsilon f(\tau_0, x^\circ, y^\circ)$$

The variable $y = y^*$ can be defined from the first equation of (1.1) independently of $y^{\cdot}$, and $y^{\cdot}$ can be found explicitly from the second.

In what follows we will consider an oscillating system of the form (1.1).

2. ***Reduction to linear form in the first approximation with respect to a small parameter.*** We will perform this procedure in two ways: 1) using the "amplitude-phase" variables r, ψ, and 2) using the oscillating variables a, b of the Van der Pol type.

1) We make the necessary change of the initial variables x, y to new ones r, ψ by means of the relations

$$(x, y) \to (r, \psi):\ x = r\cos\psi, \quad y = -r\sin\psi \ (r \neq 0) \tag{2.1}$$

As usual /2, 9/, differentiating expressions (2.1), by virtue of system (1.1) we obtain equations (the Cauchy problem) for the slow variable, the amplitude r, and the fast variable, the phase ψ, of "standard form with rotating phase"

$$r^{\cdot} = \varepsilon R(\tau, r, \psi), \quad r(0) = r^\circ \equiv (x^{\circ 2} + y^{\circ 2})^{1/2} > 0 \tag{2.2}$$

$$\psi^{\cdot} = \nu(\tau) + \varepsilon \Psi(\tau, r, \psi), \quad \psi(0) = \psi^\circ$$

$$(\cos\psi^\circ = x^\circ/r^\circ, \ \sin\psi^\circ = -y^\circ/r^\circ)$$

$$R(\tau, r, \psi) \equiv f(\tau, x, y)\cos\psi - g(\tau, x, y)\sin\psi$$

$$\Psi(\tau, r, \psi) \equiv -r^{-1}[f(\tau, x, y)\sin\psi + g(\tau, x, y)\cos\psi]$$

We understand here that the variables x, y in the functions f, g are replaced in accordance with (2.1). Next, system (2.2) is placed in correspondence with the averaged system of the first approximation (averaged over ψ) /1-3, 9, 10/:

$$\rho^{\cdot} = \varepsilon R_0(\tau, \rho), \quad \rho(0) = r^\circ \quad (R_0(\tau, r) \equiv \langle R(\tau, r, \psi)\rangle_\psi). \tag{2.3}$$

$$\varphi^{\cdot} = \nu(\tau) + \varepsilon\Psi_0(\tau, \rho), \quad \varphi(0) = \psi^\circ \quad (\Psi_0(\tau, r) \equiv \langle \Psi(\tau, r, \psi)\rangle_\psi)$$

The angle brackets denote averaging over the phase ψ.

In the system (2.3) obtained the slow variable ρ is separated from the averaged phase φ. After integrating the first equation for the amplitude ρ, in analytically or numerically, which can be done with respect to the slow time τ ($\rho' = d\rho/d\tau = R_0(\tau, \rho), \rho(\tau_0) = r^\circ$), the phase φ can be found by quadrature. Thus, assuming that we know the solution for ρ, we have

$$\rho = \rho^*(\tau, \tau_0, r^\circ), \quad \varphi = \varphi^*(\tau, \tau_0, r^\circ, \psi^\circ, \varepsilon) \equiv \psi^\circ + \theta + \varphi_0 \tag{2.4}$$

$$\theta = \frac{1}{\varepsilon}\int_{\tau_0}^{\tau} \nu(\sigma)\, d\sigma, \quad \varphi_0 = \int_{\tau_0}^{\tau} \Psi_0(\sigma, \rho^*(\sigma, \tau_0, r^\circ))\, d\sigma$$

When the Lipschitz condition or the condition of differentiability of the functions R, Ψ with respect to r, ψ, are satisfied i.e., of the functions f, g, in x, y, respectively, between the solution

$$r = r^*(t, \tau_0, r^\circ, \psi^\circ, \varepsilon), \quad \psi = \psi^*(t, \tau_0, r^\circ, \psi^\circ, \varepsilon) \tag{2.5}$$

of systems (2.2) and the solution (2.4) of system (2.3), we have ε-proximity in the asymptotically long time interval considered (see (1.1))

$$|r^* - \rho^*| + |\psi^* - \varphi^*| \leqslant c\varepsilon, \quad t \in [0, \Theta\varepsilon^{-1}], \quad c = \text{const} \tag{2.6}$$

From (2.6) and (2.1) it follows that the property of ε-proximity also occurs for unknown initial variables x, y and known approximate variables x_1, y_1

$$|x^* - x_1^*| + |y^* - y_1^*| \leqslant C\varepsilon, \quad t \in [0, \Theta\varepsilon^{-1}], \quad C = \text{const}$$
$$x = x^* \equiv r^* \cos\psi^*, \quad y = y^* \equiv -r^* \sin\psi^*$$
$$x_1 = x_1^* \equiv \rho^* \cos\varphi^*, \quad y_1 = y_1^* \equiv -\rho^* \sin\varphi^*$$

We note some analytic properties of system (2.3) with respect to the variable ρ for sufficiently small $\rho > 0$. Since, by hypothesis, the functions f, g satisfy the Lipschitz condition or are differentiable with respect to x, y in the neighbourhood of $(0, 0)$: $x^2 + y^2 \leqslant d^2$, their Fourier coefficients in the variable ψ (after substituting for x, y the expressions (2.1)) satisfy the estimates

$$|k_0(\tau, r)| \leqslant D, \quad |k_n^{c,s}(\tau, r)| \leqslant Dr; \; k = f, g \tag{2.7}$$
$$D = \text{const}, \quad \tau - \tau_0 \in [0, \Theta], \quad 0 \leqslant r \leqslant d, \; n = 1, 2, \ldots$$

For the averaged functions R_0, Ψ_0 we have analogous estimates

$$|R_0(\tau, \rho)| \leqslant D\rho, \quad R_0(\tau, \rho) \equiv {}^1/_2 [f_1^c(\tau, \rho) - g_1^s(\tau, \rho)] \tag{2.8}$$
$$|\Psi_0(\tau, \rho)| \leqslant D, \quad \Psi_0(\tau, \rho) \equiv -{}^1/_2\rho^{-1} [f_1^s(\tau, \rho) + g_1^c(\tau, \rho)]$$

which allow us to assert that system (2.3) satisfies existence and uniqueness conditions of solution (2.4).

We will now introduce new variables u, v connected with the variables ρ, φ by the relations analogous to (2.1)

$$u = \rho\cos\varphi, \quad v = -\rho\sin\varphi \quad (\rho \geqslant 0) \tag{2.9}$$

Differentiating expressions (2.9) with respect to t using Eqs.(2.3) and eliminating the functions $\cos\varphi, \sin\varphi$ according to (2.9) we obtain non-linear equations for u, v of the definite structure

$$u^{\cdot} = \varepsilon pu + (\nu + \varepsilon q) v, \quad u(0) = x^\circ; \quad v^{\cdot} = -(\nu + \varepsilon q) u + \varepsilon pv, \quad v(0) = y^\circ \tag{2.10}$$
$$\rho \equiv (u^2 + v^2)^{1/2}, \quad p = p(\tau, \rho) \equiv \rho^{-1}R_0(\tau, \rho), \quad q = q(\tau, \rho) \equiv \Psi_0(\tau, \rho)$$

(here and henceforth $\nu = \nu(\tau)$).

We substitute the known expression for $\rho = \rho^*(\tau, \tau_0, r^\circ)$ (2.4) into the expressions for the coefficients p, q in (2.10). We obtain the desired linear homogeneous system with slowly varying parameters also of definite structure, characterized by two coefficients εp^* and $\nu + \varepsilon q^*$:

$$u_1^{\cdot} = \varepsilon p^* u_1 + (\nu + \varepsilon q^*) v_1, \quad u_1(0) = x^\circ \tag{2.11}$$
$$v_1^{\cdot} = -(\nu + \varepsilon q^*) u_1 + \varepsilon p^* v_1, \quad v_1(0) = y^\circ$$
$$p^* = p^*(\tau, \tau_0, r^\circ) \equiv R_0(\tau, \rho^*)/\rho^*, \quad q^* = q^*(\tau, \tau_0, r^\circ) \equiv \Psi_0(\tau, \rho^*), \quad \rho^* = \rho^*(\tau, \tau_0, r^\circ)$$

It can be established that the solution $u = u^* \equiv \rho^*\cos\varphi^*$, $v = v^* \equiv -\rho^*\sin\varphi^*$ of Cauchy problem (2.10) is identical with the solution $u_1 = u_1^*$, $v_1 = v_1^*$ of problem (2.11).

Indeed, we pass in system (2.11) to the "amplitude-phase" variables ρ_1, φ_1 by formulae (2.9) and we obtain the equations $\rho_1^{\cdot} = \varepsilon p^*\rho_1$, $\varphi_1^{\cdot} = \nu + \varepsilon q^*$ and the conditions $\rho_1(0) = r^\circ$, $\varphi_1(0) = \psi^\circ$. Since $\rho^{*\cdot} = \varepsilon R_0(\tau, \rho^*)$, taking into account the expression for p^* in (2.11) we obtain the relation $d\rho_1/\rho_1 = d\rho^*/\rho^*$ from which it follows that

$$\rho_1 = \rho_1^* \equiv r^\circ \exp\left[\int_{\tau_0}^{\tau} p^*(\sigma, \tau_0, r^\circ)\, d\sigma\right] \equiv \rho^*(\tau, \tau_0, r^\circ) \tag{2.12}$$

and also $\varphi_1 = \varphi_1^* \equiv \varphi^*$ (see (2.4)).

Note that the (2x2) matrix of the linear system is a linear combination of the unit matrix I and a simplectic matrix J, i.e., the sum $\varepsilon p^* I + (\nu + \varepsilon q^*) J$. The coefficient εp^* defines a small dissipation (a slow change of the averaged amplitude ρ of the oscillations), and the coefficient $(\nu + \varepsilon q^*)$ defines the frequency (the rate of change of the averaged phase

φ). Characteristic exponents corresponding to the "frozen coefficients" (fixed τ) are:

$$\lambda_{1,2}(\tau, \varepsilon) = \varepsilon p^*(\tau, \tau_0, r^\circ) \pm i[\nu + \varepsilon q^*(\tau, \tau_0, r^\circ)] \tag{2.13}$$

Next, the linear Cauchy problem (2.11) also describes the initial problem (1.1) with error $O(\varepsilon)$ in the asymptotically long time interval considered $t \in [0, \Theta\varepsilon^{-1}]$ in the case when the initial conditions are given with error $O(\varepsilon)$: $u_1(0) = x^\circ + O(\varepsilon)$, $v(0) = y^\circ + O(\varepsilon)$ /9/.

We note that the approach presented above is not always convenient from the computational point of view since the expression for Ψ and Ψ_0 have singularities if the variables r and ρ change conssiderably and can take small values $(r, \rho \sim \varepsilon)$. To construct the equivalent linear system in that case, a preliminary change to osculating variables a, b of the Van der Pol type is preferable /2, 4, 7, 9/.

2) We will now change from the variables (x, y) to the variables (a, b) by a non-singular change, i.e. a linear transformation of "rotation",

$$(x, y) \xrightarrow[\theta]{} (a, b): \; x = a\cos\theta + b\sin\theta, \quad y = -a\sin\theta + b\cos\theta \tag{2.14}$$

The variable θ is defined in (2.4). We note that, as a result of relations (2.1), (2.14) between the variables r, ψ and a, b, we have the connection

$$\begin{gathered} a = r\cos(\psi - \theta), \quad b = -r\sin(\psi - \theta); \quad r^2 = a^2 + b^2 \\ \cos\psi = \cos(\theta - \delta), \quad \sin\psi = \sin(\theta - \delta) \\ \cos\delta = ar^{-1}, \quad \sin\delta = br^{-1} \quad (\psi = \theta - \delta, \mod 2\pi) \end{gathered} \tag{2.15}$$

A routine procedure leads to the Cauchy problem for the osculating variables a, b:

$$\begin{gathered} a^\cdot = \varepsilon A(\tau, a, b, \theta), \quad A \equiv f\cos\theta - g\sin\theta, \quad a(0) = a^\circ = x^\circ \\ b^\cdot = \varepsilon B(\tau, a, b, \theta), \quad B \equiv f\sin\theta + g\cos\theta, \quad b(0) = b^\circ = y^\circ \end{gathered} \tag{2.16}$$

Here expressions (2.14) for x, y are represented in the functions f, g; θ is a known function of t, τ_0, ε (or $\tau, \tau_0, \varepsilon$) playing the role of rotating phase. The standard system (2.16) (by a change of argument it can be reduced to a standard system in the sense of Bogolyubov /1-3, 9/, see below) corresponds to the system of the first approximation averaged over θ

$$\begin{gathered} \gamma' = C_0, \quad \gamma(\tau_0) = c^\circ; \quad C_0 = C_0(\tau, \alpha, \beta) \equiv \\ \langle C(\tau, \alpha, \beta, \theta)\rangle_\theta, \quad |C_0| \leqslant D(|\alpha| + |\beta|) \\ (\gamma = \alpha, \beta; \; c = a, b; \; C = A, B; \quad ' \equiv d/d\tau) \\ A_0 = {}^1/_2 [f_1^c(\tau, \alpha, \beta) - g_1^s(\tau, \alpha, \beta)], \\ B_0 = {}^1/_2 [f_1^s(\tau, \alpha, \beta) + g_1^s(\tau, \alpha, \beta)] \end{gathered} \tag{2.17}$$

Here $f_1^{c,s}$, $g_1^{c,s}$ are the first Fourier coefficients of the functions f and g, 2π-periodic in θ (after replacing x and y according to (2.14)), see (2.8) and (2.9); for example, $f_1^c = 2\langle f\cos\theta\rangle_\theta$; we define f_1^s and $g_1^{c,s}$ analogously. Unlike system (2.3), Eqs.(2.17) for α, β turn out to be connected. However, in accordance with relations (2.15) they can be reduced to the form of one equation for ρ, which has to be integrated, and the quadrature for the other variable, for example, δ (see (2.4)).

Next, the solution of the averaged Cauchy problem (2.17) can be obtained in slow time τ, $\tau - \tau_0 \in [0, \Theta]$, in a relatively short range of change of the argument by analytic or numerical methods. We shall assume that it is known

$$\alpha = \alpha^*(\tau, \tau_0, a^\circ, b^\circ), \quad \beta = \beta^*(\tau, \tau_0, a^\circ, b^\circ) \tag{2.18}$$

From the variables α, β described by system (2.17) we change to the variables u, v by formulae analogous to (2.14) and after differentiating u, v, by virtue of system (2.17) we eliminate the expression $\cos\theta$ and $\sin\theta$. We obtain a non-linear system analogous to (2.10),

$$\begin{gathered} u^\cdot = \varepsilon pu + (\nu + \varepsilon q)v, \quad u(0) = a^\circ = x^\circ \\ v^\cdot = -(\nu + \varepsilon q)u + \varepsilon pv, \quad v(0) = b^\circ = y^\circ \\ p = p(\tau, \alpha, \beta) = \frac{\alpha A_0 + \beta B_0}{\alpha^2 + \beta^2}, \quad q = q(\tau, \alpha, \beta) = \frac{\beta A_0 - \alpha B_0}{\alpha^2 + \beta^2} \\ (u = \alpha\cos\theta + \beta\sin\theta, \quad v = -\alpha\sin\theta + \beta\cos\theta) \end{gathered} \tag{2.19}$$

Here the variables α, β should be expressed in terms of u, v and θ in accordance with

(2.19) by reverse rotation: $\alpha = u\cos\theta - v\sin\theta$, $\beta = u\sin\theta + v\cos\theta$. We note that, on the basis of estimates (2.17) for A_0, B_0 the coefficients p, q, in (2.19) are bounded as $\alpha^2 + \beta^2 \to 0 \quad (u^2 + v^2 \to 0)$.

We now replace the slow variables α, β in (2.19) by the known expressions α^*, β^* (2.18). We obtain an equivalent linear homogeneous system of the form (2.11) where the coefficients

$$p^* = p(\tau, \alpha^*, \beta^*), \quad q^* = q(\tau, \alpha^*, \beta^*) \tag{2.20}$$

are known functions of the slow time τ and the parameters of the problem. The structures of the linear systems of Eqs.(2.11) with coefficients (2.11) and (2.20) respectively, are identical, i.e., their matrices are linear combinations of unit and simplectic matrices with coefficients εp^* and $\nu + \varepsilon q^*$. This property leads to the same "frozen" characteristic exponents (2.13).

Hence, we have again obtained a linear system of the form (2.11), equivalent to the initial one (1.1), with slowly varying coefficients (2.20). As we remarked above, we understand equivalence in the sense of ε-proximity of the solutions of these systems in an asymptotically long time interval $t \in [0, \Theta\varepsilon^{-1}]$ under the condition of ε-proximity of the initial conditions. If the condition of uniform asymptotic stability of the solutions (the limit points or cycles) of averaged system (2.3) or (2.17) with respect to the initial conditions is satisfied (see /2, 9/) then the proximity indicated, i.e., the equivalence of initial and linearized systems, occurs in an unbounded time interval $t \in [0, \infty)$ which is of considerable interest for analysing or synthesizing automatic control systems /5-8/.

3. Construction of the equivalent linear system with a given degree of accuracy with respect to a small parameter. We shall construct the linear system on the basis of osculating variables a, b (2.14) according to (2.16)-(2.18). The solution of system (2.16) has to be constructed with a given degree of accuracy ε^N $(N = 0, 1, 2, \ldots)$ in an asymptotically long time interval $t \in [0, \Theta\varepsilon^{-1}]$, $\Theta = \text{const}$. To this end we can use the procedures worked out, for example, in /2, 3, 7, 9-12/. We note that by introducing the argument θ by the formula $d\theta = \nu dt$, see (2.4), and the corresponding slow argument $\vartheta = \varepsilon\theta$ $(d\vartheta = \nu d\tau,\ \nu \geqslant \nu_1 > 0)$, system (2.16) can be reduced to the form $z' = \varepsilon Z(\vartheta, z, \theta)$, where the prime denotes the derivative with respect to θ, and z, Z are two-dimensional vectors, where Z is 2π-periodic in θ. Extending the vector z, $z = (z_1, z_2, z_3)$ $(z_3 \equiv \tau, Z_3 \equiv 1)$ we obtain the system $z' = \varepsilon Z(z, \theta)$ which is standard in the sense of Bogolyubov.

The indicated solution of the $(N+1)$-th approximation is constructed on the basis of the general solution α^*, β^* of (2.18), see /2, 7, 9, 12/; it has the form

$$\begin{gathered} c = c(t, \tau_0, a^\circ, b^\circ, \varepsilon) = c_N + O(\varepsilon^{N+1}) \\ c_N = c_N(\theta, \tau, \tau_0, a^\circ, b^\circ, \varepsilon) = \gamma_N(\tau, \tau_0, a^\circ, b^\circ, \varepsilon) + \\ \varepsilon\delta_N{}^c(\theta, \tau, \tau_0, a^\circ, b^\circ, \varepsilon); \quad c = a, b; \quad \gamma = \alpha, \beta \end{gathered} \tag{3.1}$$

Here γ are functions of the slow time τ, regular with respect to ε, and describe smooth (averaged) motions with error $O(\varepsilon^{N+1})$; for $N = 0$ $(\varepsilon = 0)$ they are equal to γ^* (2.18). The components $\varepsilon\delta_N{}^c$ $(c = a, b)$ are 2π-periodic in θ, they take into account small fast oscillations also with an accuracy $O(\varepsilon_N{}^c)$ and for $N = 0$ $(\varepsilon = 0)$ should be omitted. The functions γ, δ^N are again unknown.

We change from the variables a_N, b_N to the variables u, v by formulae of the type (2.19). Differentiating them with respect to t we obtain the analogous quasilinear system

$$\begin{gathered} u^\cdot = \nu v + \varepsilon(\alpha_N' + \delta_N{}^{a\cdot})\cos\theta + \varepsilon(\beta_N' + \delta_N{}^{b\cdot})\sin\theta \\ v^\cdot = -\nu u - \varepsilon(\alpha_N' + \delta_N{}^{a\cdot})\sin\theta + \varepsilon(\beta_N' + \delta_N{}^{b\cdot})\cos\theta \\ \cos\theta = (a_N u + b_N v)(a_N{}^2 + b_N{}^2)^{-1}, \quad \sin\theta = (b_N u - a_N v)(a_N{}^2 + b_N{}^2)^{-1} \end{gathered} \tag{3.2}$$

We now use the equations for the slow variables for α_N, β_N which have the form /1-3, 7, 9-11/

$$\gamma_N' = C_0(\tau, \alpha_N, \beta_N) + \varepsilon C_{(N)}(\tau, \alpha_N, \beta_N, \varepsilon); \ \gamma = \alpha, \beta; \ C = A, B \tag{3.3}$$

and we reduce the system of Eqs.(3.2) to a form little different from (2.11) and (2.20)

$$\begin{gathered} u_N{}^\cdot = \varepsilon p^* u_N + (\nu + \varepsilon q^*) v_N + \varepsilon U_N, \quad u_N(0) = u_N{}^\circ \\ v_N{}^\cdot = -(\nu + \varepsilon q^*) u_N + \varepsilon p^* v_N + \varepsilon V_N, \quad v_N(0) = v_N{}^\circ \end{gathered} \tag{3.4}$$

$$s^* = s^*(\tau, \tau_0, a^\circ, b^\circ), \quad s = p, q; \ W_N = W_N(\tau, \theta, \tau_0, a^\circ, b^\circ, \varepsilon), \quad W = U, V$$

Here εU_N and εV_N are small external actions 2π-periodic in θ defined by α_N, β_N, $A_{(N)}, B_{(N)}$, and $\delta_N{}^{a\cdot}, \delta_N{}^{b\cdot}$. This excitation, to a first approximation in ε, is not perturbed by slow variables α, β. Hence, the linear system (3.4) constructed, which is equivalent to (1.1), contains external periodic excitation and approximates the solution $x^* (t, \tau_0, x^\circ, y^\circ, \varepsilon)$, $y^* (t, \tau_0, x^\circ, y^\circ, \varepsilon)$ with error $O(\varepsilon^{N+1})$ in an asymptotically long time interval $t \in [0, \Theta\varepsilon^{-1}]$ if $u_N^\circ = x^\circ + O(\varepsilon^{N+1})$, $v_N^\circ = y^\circ + O(\varepsilon^{N+1})$.

4. Example. Consider the oscillations of a quasilinear oscillator with slowly varying parameters (1.3). Using the approach presented in Sect.2 we obtain two forms of equivalent equations of the first approximation with corresponding expressions for the coefficients p, q and p^*, q^*. According to (1.4), (2.1)-(2.3), (2.8)-(2.11) we have for the first form of representation of the coefficients (see paragraph 1 of Sect.2)

$$p = p(\tau, \rho) \equiv -\frac{\nu'}{2\nu} - \frac{h_s}{\nu\rho}, \quad p = p^*(\tau, \tau_0, r^\circ) \equiv p(\tau, \rho^*(\tau, \tau_0, r^\circ)) \tag{4.1}$$

$$q = q(\tau, \rho) \equiv -\frac{h_c}{\nu\rho}, \quad q = q^*(\tau, \tau_0, r^\circ) \equiv q(\tau, \rho^*(\tau, \tau_0, r^\circ))$$

$$h_{c,s} = h_{c,s}(\tau, \rho) = \frac{1}{2\pi}\int_0^{2\pi} h(\tau, \rho\cos\varphi, -\nu(\tau)\rho\sin\varphi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\varphi\, d\varphi, \quad r^\circ = \left(x^{\circ 2} + \frac{x^{\cdot \circ 2}}{\nu_0^2}\right)^{1/2}$$

In the second form of equations, according to (1.4), (2.14)-(2.20) (see p.2 of Sect.2) we have the analogous expressions

$$p = p(\tau, \alpha, \beta) \equiv -\frac{\nu'}{2\nu} - \frac{1}{2\nu\rho^2}(\alpha h_1^s + \beta h_1^c) \tag{4.2}$$

$$q = q(\tau, \alpha, \beta) \equiv -\frac{1}{2\nu\rho^2}(\beta h_1^s - \alpha h_1^c)$$

$$p = p^*(\tau, \tau_0, x^\circ, y^\circ) \equiv p(\tau, \alpha^*, \beta^*), \quad q = q^*(\tau, \tau_0, x^\circ, y^\circ) \equiv q(\tau, \alpha^*, \beta^*)$$

$$h_1^{c,s} = h_1^{c,s}(\tau, \alpha, \beta) = 2h^{c,s}(\tau, \alpha, \beta) = \frac{1}{\pi}\int_0^{2\pi} h(\tau, x, y)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\theta\, d\theta, \quad \rho^2 = \alpha^2 + \beta^2$$

We now reduce the equivalent systems of the first approximation (2.10), (2.11) or (2.19), (2.20) to the form of one linear "homogeneous second-order equation (the terms $O(\varepsilon^2)$ are omitted). To fix our ideas, we take the first form (2.10), (2.11), (4.1). We differentiate the first equation of (2.10) with respect to t, express $v, v^\cdot$ in terms of $u, u^\cdot, \tau$ and substitute them into the differentiated expression. We obtain the required equivalent equations with error $O(\varepsilon^2)$:

$$u^{\cdot\cdot} + \nu^2 u = -[2\varepsilon / (\nu\rho)]\,[h_s(\tau, \rho)\, u^\cdot - h_c(\tau, \rho)\, u] \tag{4.3}$$

$$u_1^{\cdot\cdot} + \nu^2 u_1 = -[2\varepsilon / (\nu\rho)]\,[h_s(\tau, \rho^*(\tau, \tau_0, r^\circ))\, u^\cdot_1 - h_c(\tau, \rho^*(\tau, \tau_0, r^\circ))\, u], \quad \rho^2 = u^2 + u^{\cdot 2} / \nu^2$$

The coefficients of $u^\cdot, u_1^\cdot$ define the effective "dissipation", and those of u, u_1 define the "returning force". Analogously, we can obtain equations of the form (4.3) on the basis of the second form of equivalent equations, see (2.19), (2.20), (4.2).

The essential difference between the equations obtained and those known in the theory of linear oscillating sytems is the fact that small non-linear peturbing terms εh can lead to dissipation and to an increase in the effective frequency (see the expresions for p, p^* and q, q^* (4.1), (4.2) for the oscillator and also general formulae (2.10), (2.11)-(2.13), (2.19), (2.20)). This property is not present in linear systems. Conversely, an increase in the amplitude of the oscillations (negative dissipation) can lead to a decrease in frequency. We note that the equation for the variable u, despite the term $\nu(\tau)v$, contains additional small components $O(\varepsilon)$ which must be taken into account in the first approximation. As was shown above, the equivalent equations of the first approximation obtained, unlike the initial one, have a definite structure (defined by two coefficients) and are also "homogeneous", i.e., they have a rest point at the origin.

We note that in the case of perturbations f, g, or h that are polynomials in x, y or $x, x^\cdot$ the construction of equivalent equations leads to elementary quadratures of integer powers of trigonometric functions of the form $\cos^n\varphi\sin^m\varphi$ or $\cos^n\theta\sin^m\theta$ in the interval $[0, 2\pi]$ and also to the integration of averaged equations of the first approximation (2.3), (2.17) which may involve some computational costs. If the averaged equations do not depend on τ (for ρ or α, β) then the integration is performed in quadratures; we can also use here methods of integrating equations of the first order. Classical equations of the type of Duffing, Van der Pol and many others /1-4, 9, 11/ can be investigated to the end. The approach also turns out to be fairly effective in more-complicated cases of non-linearity: of the dry friction type, air gap, hysteresis loops, quadratic friction etc. /2-6, 11/. Of course, the case of non-smooth perturbations requires appropriate justification.

The results obtained can be directly carried out to systems of the form (1.1) or (1.3) containing, in addition, a slow vector z: $f = f(\tau, x, y, z)$, $g = g(\tau, x, y, z)$ or $h = h(\tau, x, x^{\cdot}, z)$, where $z^{\cdot} = \varepsilon Z(\tau, x, y, z)$ or $z^{\cdot} = \varepsilon Z(\tau, x, x^{\cdot}, z)$. Here, however, $\nu = \nu(\tau)$, i.e., the dependence of ν on z cannot be permitted since it can lead to so called essentially non-linear systems, which require additional study.

The development and justification of approaches to constructing equivalent linear systems for integrodifferential equations, for essentially non-linear systems, and also for multi-particle (multiphase) systems, are of considerable theoretical and practical interest.

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